

# Prospects for Charged Particle Astronomy

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## Abstract

The likelihood of detecting individual discrete sources of cosmic rays depends on the mean separation between sources. The analysis here derives the minimum separation that makes it likely that the closest source is detectable. For super-GZK energies, detection is signal limited and magnetic fields should not matter. For sub-GZK energies, detection is background limited, and intergalactic magnetic fields enter the analysis through one adjustable parameter. Both super-GZK and sub-GZK results are presented for four different types of sources: steady isotropic sources, steady jet sources, isotropic bursts, and jet bursts.

Keywords: cosmic ray; anisotropy; propagation; discrete source

## 1 Introduction

The rapid growth of the Pierre Auger Cosmic Ray Observatory's data set has prompted speculation that one or more discrete sources should soon be detected at ultra-high cosmic ray energies. Just as nearby stars are easily detected against the night sky, any nearby cosmic ray source should manifest itself as a cluster of cosmic ray arrival directions in the sky. At sufficiently high energies, magnetic deflections of charged particles from nearby sources should be small, so tight clustering of those arrival directions can be expected. At the highest energies, all of the observed cosmic rays should be coming from nearby sources because pion photoproduction (the GZK effect) eliminates high energy protons from distant sources.

It is presently not possible to predict with certainty whether or not new observatories (e.g. Auger [1], Telescope Array [2]) will identify and study individual discrete cosmic ray sources. The nature of the primary particles is still unknown and so is the mechanism that endows them with extremely high energy. Even if it is assumed that the highest energy cosmic rays are protons produced by discrete extragalactic sources, those sources might be distributed densely like galaxies or widely separated like the most powerful quasars. The mean separation between sources is crucial in evaluating the probability that the nearest source is detectable.

Assumptions about intergalactic magnetic field properties are also critical in evaluating this probability. Proton trajectories are bent by magnetic fields, so the flux of particles from a discrete source is spread over a solid angle of the sky. The amount of blurring due

to intervening extragalactic fields depends on the distance to the source, on the typical magnetic field strength in intergalactic space, and on the effective coherence length for magnetic field directions. Properties of intergalactic magnetic fields are highly uncertain [3, 4, 5]. The better-known magnetic fields *within* the Galaxy will blur any extragalactic source somewhat, but at the highest energies the blurring of an extragalactic proton source due to the Galaxy's fields is less than the detector's point spread function (cf. Appendix B).

Prospects for charged particle astronomy may be greatest above the GZK energy threshold for pion photoproduction [6]. Protons that are now detected well above the GZK energy threshold must have been produced within the last few hundred million years. Their sources are necessarily close enough, and their magnetic rigidities are great enough, to expect that their arrival directions must point close to the celestial positions of their sources. Moreover, the GZK effect eliminates the isotropic background which, at energies below the GZK threshold, has been accumulating for roughly 5 billion years.

It is important to note that the favorable prospects for super-GZK charged particle astronomy stem from two independent considerations. It is a coincidence that magnetic blurring by galactic magnetic fields and by plausible extragalactic fields becomes unimportant for protons approximately at the threshold energy where the GZK effect erases the isotropic background. Magnetic fields have nothing to do with pion photoproduction. If the GZK energy threshold were lower, charged particle astronomy would be favorable above the threshold even though intervening magnetic fields might spread any one source over a large solid angle on the sky, simply because the background would be gone. Moreover, if magnetic fields were negligible below the GZK threshold, point source excesses would be detectable despite the isotropic background. With the two effects taking hold at approximately the same energy, it is assured that super-GZK astronomy is signal-limited, whereas sub-GZK charged particle astronomy is background-limited.

Neither the GZK erasure of isotropic background nor the defeat of magnetic blurring by high rigidity has a sharp onset with energy. Instead of designating any particular energy as the dividing line between sub-GZK and super-GZK particle astronomy, this paper will rely on the characterization of super-GZK as being signal-limited and sub-GZK as background limited. Signal-limited analysis will be called super-GZK, whereas background-limited analysis will be called sub-GZK.

The basic model to be used throughout most of this paper assumes that ultra-high energy cosmic rays are produced by a set of sources with mean separation  $R$  and that all of them have the same luminosity  $Q$  (cosmic rays per unit time). The density of cosmic rays in the universe today results from that emission rate density over an accumulation time  $T$  (which is limited by electron pair production below the GZK threshold energy and by pion photoproduction above the threshold). The individual source luminosity  $Q$  can be calculated from  $R$ ,  $T$ , and the observed cosmic ray intensity  $I$ . For any assumed distance to the nearest such source, the expected flux is then determined. Its detectability at that distance depends simply on the cumulative detector exposure and, in the case of sub-GZK astronomy, how much solid angle is spanned by the magnetically blurred flux.

It is easy to understand that the nearest source is more likely to be detectable if the sources are widely separated. Today's observed cosmic ray density is  $Q \times T/R^3$  (the density of sources being  $1/R^3$ ), so each source must have luminosity  $Q \propto R^3$  in this

model. A larger mean separation  $R$  requires a greater luminosity  $Q$  per source to account for the observed cosmic ray density. The flux from a single source at distance  $r$  will be  $Q/4\pi r^2 \propto R^3/r^2$ . Since the nearest source is likely to be at distance  $r \sim R$ , the expected flux from the nearest source grows linearly with  $R$ . In the super-GZK (signal-limited) regime, this suffices to show that sources are more detectable if the mean separation of sources  $R$  is large. (This result pertains only when  $R$  is less than the GZK attenuation length.) For the sub-GZK case, one must worry about magnetic fields enlarging the solid angle over which the signal is seen by the detector. If the magnetic smearing is stochastic, that solid angle grows linearly with  $r$  (and hence  $R$ ), but the noise (fluctuation of the background within that solid angle) grows only as its square root. The signal-to-noise (detectability criterion) is therefore enhanced by large values of  $R$  also in the sub-GZK regime.

The analysis proceeds to find the minimum separation  $R$  that is needed in order for there to be greater than a 50-50 chance that the nearest source will be detectable. The answer depends especially on the assumed cosmic ray accumulation time  $T$  for particles above an adopted energy cut and on the number of arrival directions in the detector's data set that are above that energy cut. In the background-limited case, the answer also depends on a parameter that characterizes small-angle magnetic scattering in intergalactic space. For any set of assumed values for these parameters, one obtains the minimum mean separation of sources for which the detection of discrete sources is likely.

The basic model assumes that the different sources of ultra-high energy cosmic rays are identical. They persist in time and emit cosmic rays isotropically. The model can be modified to study sources that emit in collimated jets, or transient sources that emit their cosmic rays isotropically in a burst, or bursts with jets. Part of the analysis can be performed with an arbitrary luminosity function rather than assuming that all sources have the same luminosity.

No cosmic ray observatory has so far achieved full-sky coverage. Partial sky coverage is an obvious handicap in detecting discrete sources, for the detector may not get exposed to the brightest source. In addition, the analysis becomes more complicated without full-sky exposure. Section 10 indicates the necessary modifications for realistic calculations with partial sky coverage. The simplifying assumption in the bulk of this paper is that the detector has the same exposure to sources anywhere in a fraction of the sky denoted  $f_D$ . For a detector at one mid-latitude site, the effective fraction  $f_D$  is approximately 1/2 (cf. section 10).

## 2 Notation

- $E$  Cosmic ray energy, measured in  $EeV$  ( $1 EeV = 10^{18}eV$ ). In this paper, the energy  $E$  usually denotes the minimum energy used in the search for a discrete source. Due to the steep cosmic ray energy spectrum, most of the included cosmic rays are not much above that minimum energy.
- $Q$  The luminosity of a cosmic ray source (number of cosmic rays per unit time).
- $n$  The spatial density of cosmic ray sources (number of sources per unit volume). The meaning of  $n$  is somewhat different in the cases of bursting sources. (See the

description of  $\eta$  in this section.)

- $R$  The mean separation of cosmic ray sources ( $n = 1/R^3$ ).  $R$  is measured in  $Mpc$  ( $3.26 \times 10^6$  light years).
- $r$  The distance to a source.
- $r_0$  The distance to the *nearest* source.
- $c$  The speed of light.
- $T$  The effective time (measured in *years*) over which the cosmic rays have accumulated. This depends on energy. For energies below the GZK threshold that are considered here,  $cT$  is limited by  $e^\pm$  pair production to roughly  $1500 Mpc$ . Well above the GZK threshold,  $cT$  is roughly  $30 Mpc$ . The energy dependence of  $T$  through the threshold energy region is governed largely by the thermal spectrum of the CMB target photons [7].
- $I$  Or  $I(> E)$ . This is the integral intensity (cosmic rays per unit area per unit solid angle per unit time). “Integral intensity” means integrated over all energies above  $E$ .
- $t$  The total operating time of the observatory, measured in years.
- $A_0$  The ground area of the observatory measured in  $km^2$ . For a source at zenith angle  $\theta$ , the effective collecting area is  $A = A_0 \cos \theta$ .
- $\mathcal{E}$  The detector’s exposure to a discrete source (and its part of the sky). This exposure is measured in  $km^2 yr$ . For example, a detector array with acceptance out to  $60$  degrees zenith angle has an exposure to a source that passes through its zenith given by

$$\mathcal{E} = detector\_area \times (live\_time) \times \sqrt{3}/2\pi.$$

- $f_D$  The fraction of the sky well exposed to the detector. For an observatory with uniform full-sky exposure,  $f_D$  would be 1. For a detector at a single mid-latitude site with uniform acceptance to  $60^\circ$  zenith angle, the acceptance varies with declination, but an effective fraction  $f_D = 1/2$  is appropriate. See section 10.
- $N$  The total number of arrival directions in a cosmic ray data set from a single observatory site. For the case of a full-sky observatory with uniform celestial exposure,  $N$  is  $1/2$  the total number of events in the data set.
- $\omega$  The solid angle over which the signal events arrive from a particular source. These may be expected to be distributed as a 2-dimensional Gaussian given by width  $\sigma$ . In that case, the solid angle is given by  $\omega = 4\pi\sigma^2$ .
- $\mathcal{N}$  Noise, i.e. the amount of fluctuation in the expected background count. For high statistics, the noise is the square root of the background:

$$\mathcal{N} = \sqrt{I\omega\mathcal{E}}.$$

- $\mathcal{S}$  Signal. This might be the number of counts in a target region above an expected background count. If a source is expected to produce a Gaussian distribution of arrival directions, then the signal  $\mathcal{S}$  might be the Gaussian-weighted sum (minus the expected Gaussian-weighted sum from the isotropic background).
- $\Sigma$  Signal-to-noise detection threshold. For example, if a 5-sigma detection is required,  $\Sigma$  would be set equal to 5. It is the number of sigmas deemed necessary to qualify for a positive detection. In a prescribed single trial,  $\Sigma = 3$  might be appropriate [8].

- $K$  The number of in-target arrival directions needed for a background-free positive source detection. For example, a cluster of 5 super-GZK arrival directions might constitute persuasive evidence for a source in a full-sky survey, so  $K$  might be set to 5. For a previously identified most promising candidate source, finding even 2 or 3 new arrival directions in a small target region might be sufficient evidence. The number adopted for  $K$  depends on the search circumstances.
- $\kappa$  The angular diffusion coefficient that governs how the solid angle of magnetic blurring increases with source distance:  $\omega = \kappa r$ .
- $B$  Extragalactic magnetic field strength measured in nanogauss.
- $Z$  Electric charge of the cosmic rays, measured in units of the proton charge.
- $\rho$  Larmor radius. To a good approximation,  $\rho = E/ZB$ , where  $B$  is the transverse field strength in  $nG$ , energy  $E$  is measured in  $EeV$ ,  $Z$  is electric charge, and  $\rho$  is measured in  $Mpc$ . This formula is appropriate for estimations in intergalactic space. Within the Galaxy, the same formula can be used if  $\rho$  is measured in  $kpc$  and  $B$  is measured in  $\mu G$ . See Appendix B.
- $L$  The magnetic field coherence length measured in  $Mpc$ . This is a typical distance over which an intergalactic magnetic field affecting ultra-high energy cosmic rays can be regarded as having a consistent direction.
- $\mu$  The expected number of cosmic ray sources within a volume of specified radius. The radius is usually the distance limit at which any source is expected to be detectable.
- $\Omega_J$  The emission solid angle of a single jet (collimated emission of cosmic rays).
- $\Omega$  The solid angle on a sphere at distance  $r$  from a source over which the particles from a jet have spread.
- $\eta$  The spacetime density of bursting sources (rate of bursts per unit volume). For an accumulation time  $T$ ,  $n \equiv \eta T$  is the spatial density of bursts that contributed to the present-day cosmic rays, and  $R = 1/n^{1/3}$  is the mean separation between the relic fossils of those bursts.
- $W$  The number of cosmic rays emitted in a burst. For identical sources of bursts, this number is the same for all of them.
- $\tau$  The time interval over which cosmic ray protons from an instantaneous burst arrive at Earth. The spread in time is caused by different trajectory lengths due to intervening magnetic fields. The time interval  $\tau$  is expected to increase with distance to the source and decrease with cosmic ray energy.
- $\alpha$  The coefficient that governs how the time spread  $\tau$  of received cosmic rays from a burst source increases with distance to the source:  $\tau = \alpha r$  (cf. Appendix C).
- $\xi$  Used in Appendix A,  $\xi$  is a dimensionless measure of distance given by  $\xi = r/cT$ .

### 3 The basic model

The simplest model is that the observed cosmic ray intensity  $I(> E)$  is the result of isotropic emission by sources with identical luminosity  $Q$  and spatial density  $n$  over some history of  $T$  years. The density of cosmic rays is then given in terms of the sources by  $nQT$ . That same cosmic ray density is given in terms of the observed intensity as  $4\pi I/c$ .

Equating these two expressions yields the source luminosity

$$Q = \frac{4\pi IR^3}{cT}, \quad (1)$$

where the density  $n$  has been expressed in terms of the mean source separation  $R$  such that  $n = 1/R^3$ . The flux from a source at distance  $r$  is  $Q/4\pi r^2$ , and multiplying this by the exposure  $\mathcal{E}$  gives the signal

$$\mathcal{S} = \frac{IR^3}{cTr^2} \mathcal{E}. \quad (2)$$

Exposure  $\mathcal{E}$  is the time-integrated perpendicular collecting area for flux coming from the source direction, and  $\mathcal{E}$  is measured in units of  $km^2 yr$ .

Note: The flux  $Q/4\pi r^2$  should be multiplied by  $e^{-r/cT}$  if the effective accumulation time  $T$  in equation 1 is governed by a propagation attenuation such as GZK pion photoproduction. Including this factor prevents a simple analytic solution. The factor is irrelevant for the sub-GZK analysis. Omitting the factor leads to a valid result also in the super-GZK case in typical circumstances, as explained in Appendix A.

The product  $I\mathcal{E}$  is closely related to the total number of events in the data set. That number is a convenient parameter. Consider first a single observatory site such as Auger South, and let  $N$  be the number of events above some energy cut. To be specific, suppose this data set includes those events with zenith angles less than  $60^\circ$ . For a source that passes through the zenith of the detector, it is observable  $1/3$  of the operating time and the average collecting area is

$$\langle A \rangle = \frac{3}{\pi} \int_0^{\pi/3} A_0 \cos(\theta) d\theta = \frac{3\sqrt{3}}{2\pi} A_0$$

where  $A_0$  is the full detector area and  $\theta$  is zenith angle. The exposure to the source is then  $\mathcal{E} = \frac{t}{3} \langle A \rangle = \frac{\sqrt{3}}{2\pi} A_0 t$ , where  $t$  is the cumulative detector operating time. The number of events  $N$  in the full data set is the cosmic ray intensity  $I$  times the product of aperture and operating time  $t$ . The detector's aperture (accepting events out to  $60^\circ$  zenith angle) is  $\frac{3\pi}{4} A_0$ , so  $N = \frac{3\pi}{4} A_0 I t$ . Combining this with the expression for  $\mathcal{E}$  yields

$$I\mathcal{E} = \frac{2\sqrt{3}}{3\pi^2} N = 0.117 N.$$

For a full-sky observatory (a second site in the other hemisphere), the exposure to this source would be little changed, but the number of events in the data set would be twice as great. Using the approximation that a two-site observatory has uniform celestial exposure [9], the calculation for the one source here would apply to all sources. The rule to be adopted, therefore, is that  $I\mathcal{E} = 0.117 N$ , where  $N$  is the number of events in the data set for a single-site observatory or half the number of events in the data set of a full-sky observatory. (See section 10 for further discussion of non-uniform celestial exposure.)

Super-GZK analysis:



Suppose  $K$  signal showers are needed to make a positive source detection in a signal-limited regime. Substituting  $K$  for  $S$  in equation 2 gives

$$K = \frac{I\mathcal{E}R^3}{cTr^2} \Rightarrow r_K^2 = \frac{I\mathcal{E}R^3}{cTK} \quad (3)$$

where  $r_K$  is the distance at which the expected signal  $\mathcal{S}$  would be equal to  $K$ . A source is detectable within that radius around us if it is in the fraction  $f_D$  of the celestial sphere. The expected number  $\mu$  of detectable sources is that exposed volume times the source density  $1/R^3$ :

$$\mu = \frac{4\pi f_D}{3} r_K^3 / R^3.$$

Substituting the previous expression for  $r_K$ , this becomes

$$\mu = \frac{4\pi f_D}{3} \left( \frac{I\mathcal{E}R}{cTK} \right)^{3/2}. \quad (4)$$

Denote by  $r_0$  the distance to the nearest source. What condition on  $R$  ensures that  $r_0$  is likely to be less than  $r_K$ ? The probability that  $r_0$  is less than  $r_K$  is the complement of the Poisson probability that there are 0 sources within that volume when the expected number is  $\mu$ :

$$P(r_0 < r_K) = 1 - \text{Pois}(0; \mu) = 1 - e^{-\mu}.$$

Setting this final expression to be greater than 1/2 (i.e. the condition that detection is more likely than not), gives the condition  $\mu > \ln(2)$ . Then substituting the above formula for  $\mu$  yields this lower limit for the mean separation of sources:

$$R > \left( \frac{3\ln 2}{4\pi f_D} \right)^{2/3} \frac{cTK}{I\mathcal{E}}. \quad (5)$$

Alternatively, using  $I\mathcal{E} = 0.117N$  and  $f_D = 1/2$ , the minimum separation  $R_{min}$  for which the nearest source is likely to be detected is given by

$$\boxed{R_{min} = 4.1 \frac{K}{N} cT.} \quad (6)$$

This equation quantifies the conditions for detectability in the signal-limited regime. The required minimum separation  $R_{min}$  shrinks for any energy cut (fixed  $T$ ) as the number of arrival directions  $N$  increases (e.g. as an observatory's exposure increases over time).

The expression for  $r_K$  in equation 3 is derived without including GZK flux attenuation. The resulting expressions for  $\mu$  and  $R$  are valid only if  $r_K$  is much smaller than  $cT$ . As discussed in Appendix A, this is usually satisfied. The above expression for  $R_{min}$  is therefore normally justified. There is also an *upper* limit on  $R$ , since the nearest source should not be much farther away than the attenuation length  $cT$ . Appendix A shows how to find this upper limit  $R_{max}$  for which detection of the nearest source is likely.

Sub-GZK analysis:

The signal from a single source is presumed to be spread over a solid angle  $\omega$  by virtue of small magnetic bends while passing through intergalactic space with some unknown spectrum of magnetic field strengths and randomly changing directions. By invoking the central limit theorem, or by analogy with multiple Coulomb scattering, it is natural to postulate that this magnetic blurring increases linearly with distance:  $\omega = \kappa r$ . This relation defines the diffusion coefficient  $\kappa$ . (The RMS angle in any plane that contains the central direction to the source increases with the square root of the distance traveled. The solid angle increases linearly with distance because it is proportional to the product of two such angles.)

The expected background within the solid angle  $\omega$  is  $I\omega\mathcal{E}$ . The noise is the fluctuation in this background. For large statistics, that is the square root of the background, so

$$\mathcal{N} = \sqrt{I\kappa r\mathcal{E}}.$$

Using the expression for  $\mathcal{S}$  above, the signal-to-noise ratio is

$$\mathcal{S}/\mathcal{N} = \sqrt{\frac{I\mathcal{E}}{\kappa}} \frac{R^3}{cT r^{5/2}}.$$

Now suppose a detection requires that  $\mathcal{S}/\mathcal{N} > \Sigma$  for some number  $\Sigma$ , and denote by  $r_\Sigma$  the distance at which this signal-to-noise ratio occurs. The equation above gives

$$r_\Sigma^{5/2} = \sqrt{\frac{I\mathcal{E}}{\kappa}} \frac{R^3}{cT\Sigma}. \quad (7)$$

A source is detectable within the radius  $r_\Sigma$  around us. The expected number of sources within that radius is the volume times the source density:

$$\mu = \frac{4\pi f_D}{3} r_\Sigma^3 / R^3 \quad \Leftrightarrow \quad \mu = \frac{4\pi f_D}{3} \left(\frac{I\mathcal{E}R}{\kappa}\right)^{3/5} \left(\frac{1}{cT\Sigma}\right)^{6/5}. \quad (8)$$

Denoting by  $r_0$  the distance to the nearest source, as above, detection of the nearest source is likely if the probability is greater than 1/2 that  $r_0$  is less than  $r_\Sigma$ . That probability is given by the complement of the Poisson probability that there are 0 sources within that volume when the expected number is  $\mu$ . As above, this means  $\mu > \ln(2)$ , which here reduces to

$$R > \left(\frac{3\ln 2}{4\pi f_D}\right)^{5/3} (cT\Sigma)^2 \frac{\kappa}{I\mathcal{E}}. \quad (9)$$

This is the constraint on the mean source separation  $R$  such that a detection of significance  $\Sigma$  is likely.

Using  $I\mathcal{E} = 0.117N$  and  $f_D = 1/2$ , the minimum source separation for which it is likely that the nearest source is detectable becomes

$$\boxed{R_{min} = 1.4 (cT \Sigma)^2 \frac{\kappa}{N}}. \quad (10)$$

The smearing by magnetic deflection is presumed to be the result of many small-angle scatterings as a charged particle passes through irregular magnetic fields between



the source and the Earth. The process is mathematically similar to multiple Coulomb scattering, and a Gaussian distribution of arrival directions can be expected. In that case, the solid angle  $\omega$  is not a simple target region. Instead, each arrival direction should be weighted in proportion to the expected Gaussian distribution. The simple  $S/N$  analysis here still pertains, provided the weighting factor is taken to be  $4\pi\sigma^2$  times the Gaussian probability distribution. See Appendix D for further details.

## 4 Fiducial calculations

The inequalities 5, 9 and equations 6, 10 give expressions for the minimum value of the mean source separation  $R$  such that the nearest source is likely to be detectable. The minimum mean separations depend on variables that are not known with certainty. Here some fiducial parameter values are adopted for illustration. Readers who favor other parameter values can readily scale the answers for their values.

For the super-GZK estimate, the fiducial estimates here will be based on  $cT = 100 \text{ Mpc}$  and  $N = 50$ . There is some energy for which the accumulation time is  $(100 \text{ Mpc})/c$ , and any observatory should eventually detect 50 air showers above that energy. For  $K$ , it will be assumed here that there is reason to suspect the existence of a source at a particular location and that a single-trial test has been prescribed. A cluster of  $K = 3$  arrival directions at that celestial position would be a strong positive result if there are only 50 events in a data set covering much of the sky.

With these adopted values for the parameters  $K$ ,  $cT$ , and  $N$ , equation 6 gives  $R_{min} \approx 25 \text{ Mpc}$ . The nearest source is likely to be detectable if the sources are separated by more than  $25 \text{ Mpc}$  on average.

A fiducial calculation for the sub-GZK case requires adopting a value for  $\kappa$ , the coefficient that controls how magnetic blurring increases with source distance. Wild guesses are allowed here as the properties of extragalactic magnetic fields are poorly determined by observations or theory [3]. One simplistic model is that the magnetic field has a typical strength  $B$  that is randomly oriented, but a particle experiences the same orientation for coherence length  $L$ . Its total path of length  $r$  is made up of  $r/L$  deflections from these randomly oriented fields. Appendix B shows that, in this simplistic model,  $\kappa$  is given by

$$\kappa = \frac{4\pi}{9} LZ^2 B^2 / E^2. \quad (11)$$

If  $L = 1 \text{ Mpc}$ ,  $B = 1 \text{ nG}$ ,  $Z = 1$ , and  $E = 10 \text{ EeV}$ , then  $\kappa = \frac{\pi}{900} = 0.014 \text{ sr/Mpc}$ .

As an example, suppose  $cT = 1500 \text{ Mpc}$  (a typical survival distance for nucleons against pair production at energies above the spectrum's ankle), and let  $N = 10^4$  arrival directions. As in the super-GZK fiducial estimate, suppose a prescribed test has been applied to a suspected celestial location, so  $\Sigma = 3$  is statistically significant (a 3-sigma result for a single trial). Using  $cT = 1500 \text{ Mpc}$ ,  $N = 10,000$ ,  $\Sigma = 3$ , and  $\kappa = 0.014$ , equation 10 gives  $R_{min} = 40 \text{ Mpc}$ . The nearest source is expected to be detectable at this sub-GZK energy provided the sources are separated by at least  $40 \text{ Mpc}$  on average.

The estimate for the detection distance  $r_\Sigma$  would not be meaningful if the magnetic blurring at that distance (and for the energy cut  $E$ ) were to produce a solid angle greater

than  $2\pi$ . In this fiducial calculation with  $\kappa = 0.014 \text{ sr}/Mpc$ , the magnetic blurring for a source at 40  $Mpc$  is 0.56  $sr$ , corresponding to a circle of 24 degrees radius on the sky.

The fiducial parameters adopted here may be quite wrong. They are presented only to illustrate the use of the formulas and provide explicit answers that can be easily scaled for different values of the parameters.

## 5 Comments about the dependences

In the super-GZK regime, the signal can be maximized by lowering the energy cut as much as possible without violating the signal-limited condition. For the actual mean source separation  $R$ , therefore, detecting the nearest source is made more probable by reducing the energy cut as much as possible in the signal-limited regime.

In the sub-GZK regime, there may be little dependence of  $R_{min}$  on the energy cut. At least for the simplistic model treated in Appendix B,  $\kappa$  depends on energy as  $1/E^2$ , which is approximately the same as the energy dependence of the intensity  $I(> E)$  or number of events  $N$ . The factor  $\frac{\kappa}{I\mathcal{E}}$  is almost independent of energy, provided the detector's acceptance (aperture) is not growing with energy. Increasing the minimum energy reduces the magnetic blurring, but the effect on  $S/N$  is offset by the lower statistics. (There may be some advantage in raising the minimum energy in the analysis if the detector's acceptance does increase with energy.) Near the GZK threshold, energy dependence enters also through the accumulation time  $T$ . Raising the minimum energy in the analysis would then reduce  $R_{min}$ .

The statistical criterion for detection encoded in  $\Sigma$  enters quadratically in the inequality 9 and equation 10. As noted above, this number can be made relatively small if a careful sky survey determines a single best candidate source, its solid angle extent, and an optimal energy cut. A 3-sigma result for a single test with new data would then be a compelling result.

Intergalactic magnetic fields might someday be studied using discrete sources of cosmic rays at known distances. For now, gross uncertainty is encoded in the single coefficient  $\kappa$ , and any fiducial calculation based on an adhoc value of  $\kappa$  should be treated with appropriate suspicion.

Large detector exposure  $\mathcal{E}$  is obviously crucial for detecting discrete sources. Inequalities 5 and 9 show that the minimum value for the mean distance between sources in order for them to become detectable will shrink inversely as the exposure increases. A vast increase in exposure would lead to source detections, or else it would radically shrink the viable parameter space that describes the source distribution and magnetic fields. See section 10 for additional discussion about the dependence on exposure.

The expected number of detectable sources  $\mu$  grows in proportion to  $\mathcal{E}^{3/2}$  for the super-GZK case of signal-limited detection. A 10-fold increase in exposure results in a 30-fold increase in the number of detectable sources, and this is true even if that number is less than 1! Success in charged particle astronomy is all about achieving huge exposure.

The  $\mathcal{E}^{3/2}$  dependence of  $\mu$  on  $\mathcal{E}$  is closely related to the "logN-logS" relation of astronomy, in which the source count varies like  $flux^{-3/2}$  for a homogeneous distribution of detectable sources in Euclidean space, independent of their luminosity distribution.

Since  $flux = K/\mathcal{E}$  in the analysis here, the number of detectable sources increases with exposure like  $\mathcal{E}^{3/2}$ .

For background-limited (sub-GZK) analysis, the number of sources grows with exposure also, but it increases less rapidly with  $\mathcal{E}$ . The number of detectable sources is proportional to  $\mathcal{E}^{3/5}$ .

## 6 Sources with jets

Jets are a common phenomenon among objects known to produce energetic particles. Relativistic bulk plasma motion is advantageous in accelerating particles to high energies, so it is reasonable to conjecture that the highest energy cosmic rays could be emitted in collimated jets. The previous assumption of isotropic emission from cosmic ray sources might be inappropriate.

Suppose a single jet emits ultra-high energy cosmic rays in a solid angle  $\Omega_J$ . The luminosity of every source is calculated as before:

$$Q = \frac{4\pi IR^3}{cT}.$$

The measured signal from any one source is zero unless the observer is within the solid angle of the beam, which grows with distance  $r$  from the source by magnetic deflections. Let  $\Omega$  be the solid angle on the sphere of radius  $r$  which gets flux from the jet. Then the signal  $S$  is

$$\mathcal{S} = \frac{Q}{\Omega r^2} \mathcal{E} = \frac{4\pi}{\Omega} \frac{IR^3}{cT r^2} \mathcal{E}.$$

### Super-GZK analysis:

A simple assumption is that the magnetic rigidity of super-GZK particles is high enough that  $\Omega \approx \Omega_J$ , i.e. magnetic deflection does not significantly increase the solid angle of the jets in transit between the source and Earth. The signal  $S$  at distance  $r$  is then given by the previous equation with  $\Omega$  replaced by  $\Omega_J$ .

For a given exposure  $\mathcal{E}$  and cosmic ray intensity  $I$  accumulated over  $T$  years,  $K$  particles would be expected at distance  $r_K$ :

$$r_K^2 = \frac{4\pi}{\Omega_J} \frac{I\mathcal{E}R^3}{cTK}. \quad (12)$$

The probability is  $\Omega_J/4\pi$  that Earth is in the beam of any one source, so the density of viewable sources is  $\frac{\Omega_J}{4\pi R^3}$ . The expected number of *detectable* sources (i.e. pointing at us) within the exposed volume of radius  $r_K$  around us is this:

$$\mu = \frac{f_D \Omega_J}{3} \frac{r_K^3}{R^3} = \frac{f_D}{3\sqrt{\Omega_J}} \left( \frac{4\pi I\mathcal{E}R}{cTK} \right)^{3/2}. \quad (13)$$

As in the basic model above, the probability that the nearest source is closer than  $r_K$  is greater than  $1/2$  provided  $\mu > \ln 2$ . Using the above expression for  $\mu$  converts this inequality to a lower limit on the mean separation of sources:

$$R > \left(\frac{\Omega_J}{4\pi}\right)^{1/3} \left(\frac{3\ln 2}{4\pi f_D}\right)^{2/3} \frac{cTK}{I\mathcal{E}}. \quad (14)$$

Using  $I\mathcal{E} = 0.117N$  and  $f_D = 1/2$  as before, this becomes

$$\boxed{R_{min} = 4.1 \left(\frac{\Omega_J}{4\pi}\right)^{1/3} \frac{K}{N} cT.} \quad (15)$$

Detection of the nearest source is more likely than not provided the sources have at least this mean separation. Notice that the minimum separation is decreased relative to the basic model by the cube root of the jet opening solid angle (as a fraction of the isotropic  $4\pi$  solid angle).

A fiducial value of  $\Omega_J$  will here be taken to be  $0.01 \text{ sr}$ , corresponding to a jet opening cone of  $3.2^\circ$  half-angle. Together with the previously adopted fiducial values ( $cT = 100 \text{ Mpc}$ ,  $N = 50$ ,  $K = 3$ ), the fiducial calculation here gives  $R_{min} = 2.3 \text{ Mpc}$ , suggesting that the nearest source should be detectable even if the sources are distributed more densely than normal galaxies.

The minimum separation  $R_{min}$  is substantially smaller than in the basic model because the signal is strong when looking into a jet. The total luminosity of every source can be relatively weak (as must be the case if the density of sources is high), and an individual source can nevertheless be detected far away. (For  $R_{min} = 2.3 \text{ Mpc}$  in this fiducial estimate, the maximum detection distance  $r_K$  is  $17 \text{ Mpc}$ , but that detection distance grows in proportion to  $R^{3/2}$ ).

In the super-GZK regime, jet sources are easier to detect than isotropic sources if the mean separation between sources is small, but they are more difficult to detect if the mean source separation is large. Since  $r_K/R$  is larger for jet sources than for isotropic sources, it may happen that there are many sources within the GZK volume but none of them is pointing at us. For example, suppose  $R = 25 \text{ Mpc}$  (the minimum mean separation for detectability in the fiducial estimate for the isotropic case). The expected number of sources within  $100 \text{ Mpc}$  of Earth in the exposed half of the sky would be 134. For  $\Omega_J = 0.01 \text{ sr}$ , the probability is only  $\frac{0.01}{4\pi} = 8 \times 10^{-4}$  for Earth to be in the beam of any one source. Although there are 134 sources readily detectable if any points at us, the expected number pointing at us is only 0.1. Detection is unlikely for a large mean source separation.

### Sub-GZK analysis:

Although super-GZK particles can be assumed to maintain their directions in transit from source to Earth, that is not expected in the sub-GZK regime. As sub-GZK particles get farther from the source, the effective angular extent of the jet is dominated by their magnetic deflections rather than the emission angle  $\Omega_J$  of the jet itself. Here it is therefore assumed that, at the distance to Earth, the flux from a single jet is spread over a solid

angle  $\Omega = 3\kappa r$ . (See Appendix B for an explanation of the factor 3.) The signal at distance  $r$  becomes (cf. equation 2)

$$\mathcal{S} = \frac{4\pi}{3} \frac{I\mathcal{E}}{cT\kappa} \frac{R^3}{r^3}.$$

The background does not care that the luminosity of individual sources is in jets. It is given, as in the isotropic case, by  $I\omega\mathcal{E}$ . Its square root gives the estimated fluctuation in the background. Substituting  $\kappa r$  for  $\omega$ , that noise is

$$\mathcal{N} = \sqrt{I\kappa r\mathcal{E}}.$$

The signal-to-noise is then

$$\mathcal{S}/\mathcal{N} = \frac{4\pi}{3} \sqrt{\frac{I\mathcal{E}}{\kappa^3}} \frac{R^3}{cT r^{7/2}}.$$

The analysis proceeds as in the case of isotropic sources. Adopting a value  $\Sigma$  for  $\mathcal{S}/\mathcal{N}$  gives an expression for  $r_\Sigma$ :

$$r_\Sigma^{7/2} = \frac{4\pi}{3} \sqrt{\frac{I\mathcal{E}}{\kappa^3}} \frac{R^3}{cT\Sigma}. \quad (16)$$

This is the radius at which a source should be detectable with signal-to-noise equal to  $\Sigma$ .

The expected number of detectable sources (beamed at us from the exposed fraction of sky  $f_D$ ) within the radius  $r_\Sigma$  is

$$\mu = f_D \int_0^{r_\Sigma} \Omega r^2 n dr = f_D \int_0^{r_\Sigma} 3\kappa r^3 \frac{1}{R^3} dr = f_D \frac{3}{4} \kappa r_\Sigma^4 / R^3.$$

Using the foregoing expression for  $r_\Sigma$ , the expected number of sources closer than  $r_\Sigma$  becomes

$$\mu = \frac{3f_D}{4} \left(\frac{4\pi}{3}\right)^{8/7} (I\mathcal{E})^{4/7} \left(\frac{1}{\kappa}\right)^{5/7} \left(\frac{1}{cT\Sigma}\right)^{8/7} R^{3/7}. \quad (17)$$

As in the case of isotropic sources, the probability that the nearest source is detectable (closer than  $r_\Sigma$ ) is given by  $\mu > \ln(2)$ , which here reduces to

$$R > \left(\frac{4 \ln 2}{3f_D}\right)^{7/3} \left(\frac{3}{4\pi}\right)^{8/3} \left(\frac{cT\Sigma}{\sqrt{I\mathcal{E}}}\right)^{8/3} \kappa^{5/3}. \quad (18)$$

Substituting  $0.117N$  for  $I\mathcal{E}$  and  $1/2$  for  $f_D$  as previously, this becomes

$$\boxed{R_{min} = 1.6 \left(\frac{\Sigma^2}{N}\right)^{4/3} (cT)^{8/3} \kappa^{5/3}}. \quad (19)$$

Using again the fiducial parameter values  $\Sigma = 3$ ,  $cT = 1500 \text{ Mpc}$ ,  $N = 10,000$ ,  $\kappa = 0.014 \text{ sr/Mpc}$  for sub-GZK analysis, this formula gives  $R_{min} = 33 \text{ Mpc}$ . With this minimum source separation, sources are detectable ( $\Sigma > 3$ ) out to  $r_\Sigma = 47 \text{ Mpc}$ , and the arrival directions are spread over solid angle  $\omega = 0.66 \text{ sr}$  ( $26^\circ$  cone half-angle).

The sub-GZK regime offers detection capability for jetted sources with large mean separation even though super-GZK detection might be unlikely in that case. The two regimes are complementary. Super-GZK astronomy is likely for densely distributed sources with jets, whereas sub-GZK astronomy is likely for more widely distributed sources with jets.

## 7 Isotropic burst sources

It is unlikely that the sources of high energy cosmic rays are permanent. The universe is dynamic, and even the most powerful active galactic nuclei may be temporary feeding episodes of supermassive black holes. For temporary sources, the source density  $n$  should be interpreted as the mean density of *active* sources at any time. The formulas of the preceding sections should then be appropriate.

Some modifications are needed, however, if cosmic rays are produced in brief bursts. Due to the magnetic wandering of particles en route to the Earth, the duration of flux from an instantaneous burst increases with distance from the source. The duration  $\tau$  (at any cosmic ray energy) should increase linearly with distance:  $\tau = \alpha r$  (where  $\alpha$  is energy dependent, *cf.* Appendix C). In this paper, cosmic ray sources are regarded as “bursts” if their emission lifetimes are not long compared to the time spread  $\tau$  expected at Earth for a source that emits all of its cosmic rays instantaneously. Let  $W$  be the magnitude of a burst, i.e. the total number of emitted cosmic rays. If an isotropic burst is at distance  $r$  from the Earth, the flux over time  $\tau$  is  $W/(4\pi r^2 \tau)$ . Substituting  $\alpha r$  for  $\tau$  and multiplying by the exposure  $\mathcal{E}$  gives the signal,

$$\mathcal{S} = \frac{W\mathcal{E}}{4\pi\alpha r^3}.$$

Let  $\eta$  denote the spatial density of bursts per unit time. If  $T$  is the accumulation time for cosmic rays, then  $n \equiv \eta T$  is the fossil density of bursts that have contributed to the accumulated cosmic ray density, so

$$\frac{4\pi}{c}I = nW = \eta TW.$$

Solving this for  $W$  and inserting the result into the expression for the signal  $\mathcal{S}$  gives

$$\mathcal{S} = \frac{I\mathcal{E}}{\alpha c \eta T r^3}.$$

An estimate for  $\alpha$  derived in Appendix C is

$$\alpha = \frac{L^2 Z^2 B^2}{36cE^2}.$$

### Super-GZK analysis:

As usual, suppose  $K$  events are needed for a detection:

$$\mathcal{S} = K \Rightarrow r_K^3 = \frac{I\mathcal{E}}{\alpha c \eta T K}. \quad (20)$$

The density of burst sources at distance  $r$  with flux now “on” is  $\eta\tau = \eta\alpha r$ . The number of detectable bursts closer than  $r_K$  is therefore

$$\mu = f_D \int_0^{r_K} 4\pi r^2 \eta \alpha r dr = \pi f_D \alpha \eta r_K^4.$$

Using the foregoing expression for  $r_K$  and  $R^3 = 1/(\eta T)$ , this can be written as

$$\mu = \frac{\pi f_D}{cT} \left(\frac{1}{\alpha c}\right)^{1/3} \left(\frac{I\mathcal{E}}{K}\right)^{4/3} R. \quad (21)$$

The probability  $P(r_0 < r_K)$ , that the nearest (on) source is closer than  $r_K$ , is greater than  $1/2$  (detection likely) provided

$$\mu > \ln(2) \Leftrightarrow R > \frac{\ln 2}{\pi f_D} cT (\alpha c)^{1/3} \left(\frac{K}{I\mathcal{E}}\right)^{4/3}. \quad (22)$$

Substituting  $0.117N$  for  $I\mathcal{E}$  and  $1/2$  for  $F_D$  as before, this becomes

$$\boxed{R_{min} = \frac{3.9}{f_D} (\alpha c)^{1/3} \left(\frac{K}{N}\right)^{4/3} cT.} \quad (23)$$

A fiducial calculation is obtained from the parameters used previously for super-GZK estimates together with  $\alpha c = \frac{1}{36E^2}$  (and  $E = 100 \text{ EeV}$ ). These values give  $R_{min} = 0.26 \text{ Mpc}$ . This minimum  $R$  corresponds to a maximum fossil density  $\eta T = 1/R^3 = 60/\text{Mpc}^3$  accumulated over  $T \approx 3 \times 10^8 \text{ yrs}$  ( $cT = 100 \text{ Mpc}$ ). For the average galaxy density  $0.01/\text{Mpc}^3$ , this fossil density requires a burst frequency not more than one per 50,000 years per galaxy. (The minimum mean time between bursts per galaxy is  $< \Delta t >_{min} = 0.01 R^3 T$ .)

Note, however, the strong dependence of this minimum burst interval on the size  $N$  of the data set, it being proportional to  $R^3 \sim 1/N^4$ . A five-fold increase in exposure would reduce the limit of 50,000 years down to 80 years. Detection of a discrete source would then be likely provided the mean time interval between bursts is not greater than 80 years in a typical galaxy. Increasing exposure is especially advantageous in the search for discrete sources if the sources are isotropic bursts.

#### Sub-GZK analysis:

The background noise is not changed by the assumption of burst sources. It is still given by

$$\mathcal{N} = \sqrt{I\omega\mathcal{E}} = \sqrt{I\kappa r\mathcal{E}}.$$

The signal-to-noise is therefore

$$\mathcal{S}/\mathcal{N} = \frac{I\mathcal{E}}{\alpha c \eta T r^3} / \sqrt{I\kappa r\mathcal{E}} = \sqrt{\frac{I\mathcal{E}}{\kappa}} \frac{1}{\alpha c \eta T r^{7/2}}.$$

For a specified value  $\Sigma$  for  $S/N$ , the maximum detection distance is  $r_\Sigma$ . Using  $\eta T = 1/R^3$ ,  $r_\Sigma$  is given by

$$r_\Sigma^{7/2} = \sqrt{\frac{I\mathcal{E}}{\kappa}} \frac{R^3}{\alpha c \Sigma}. \quad (24)$$

The expected number of sources closer than this distance is

$$\mu = f_D \int_0^{r_\Sigma} 4\pi r^2 \eta \tau dr = f_D \int_0^{r_\Sigma} 4\pi \alpha \eta r^3 dr = \pi f_D \alpha \eta r_\Sigma^4,$$



which, upon substituting for  $r_\Sigma$  becomes

$$\mu = f_D \frac{\pi}{cT} \left(\frac{1}{\alpha c}\right)^{1/7} \left(\frac{I\mathcal{E}}{\kappa\Sigma^2}\right)^{4/7} R^{3/7}. \quad (25)$$

Detection is likely, i.e.  $Prob(r_0 < r_\Sigma) > 1/2$ , if  $\mu > \ln(2)$ , which is

$$R > \left(\frac{\ln 2}{\pi f_D}\right)^{7/3} (cT)^{7/3} (\alpha c)^{1/3} \left(\frac{\kappa\Sigma^2}{I\mathcal{E}}\right)^{4/3}. \quad (26)$$

Using the estimate  $I\mathcal{E} = 0.117N$ , this becomes

$$\boxed{R_{min} = 0.51 \left(\frac{cT}{f_D}\right)^{7/3} (\alpha c)^{1/3} \left(\frac{\kappa\Sigma^2}{N}\right)^{4/3}.} \quad (27)$$

A fiducial estimation can be done using  $\alpha c = 1/(36E^2)$  (for  $E = 10 \text{ EeV}$ ) and the same parameters that were used for steady isotropic sources:  $cT = 1500 \text{ Mpc}$ ,  $\kappa = 0.014 \text{ sr/Mpc}$ ,  $\Sigma = 3$ ,  $N = 10,000$ ,  $f_D = 1/2$ . This gives  $R_{min} = 1.3 \text{ Mpc}$ . This minimum  $R$  corresponds to a maximum fossil density  $\eta T = 1/R^3 = 0.5/\text{Mpc}^3$  accumulated over  $T \approx 4.5 \times 10^9 \text{ yrs}$  ( $cT = 1500 \text{ Mpc}$ ). For the average galaxy density  $0.01/\text{Mpc}^3$ , this fossil density requires a burst frequency not more than one per  $9.0 \times 10^7$  years per galaxy.

As in the super-GZK case, this minimum mean time between bursts in galaxies is proportional to  $1/N^4$ . Modest increase in exposure can dramatically reduce the minimum time between bursts that would make the detection of a burst source likely.

## 8 Jet bursts

The previous two sections considered variations in which the cosmic ray sources emit in collimated jets or emit in isotropic bursts. It could also be that cosmic rays are emitted in collimated bursts. Both types of modifications to the basic model should then be incorporated together. The analysis here follows the now-familiar progression.

Each burst emits a total number  $W$  of cosmic rays above some energy threshold, and they are emitted into a solid angle  $\Omega_J$ . The spacetime density of bursts is  $\eta$  bursts per unit volume per unit time. For an accumulation time  $T$ , the spatial density of bursts that contributed to the present cosmic ray intensity is  $n \equiv \eta T$ . Thus,

$$\frac{4\pi}{c} I = nW = \eta TW \Rightarrow W = \frac{4\pi}{c} \frac{I}{\eta T}.$$

The duration of a burst at distance  $r$  from the source is  $t = \alpha r$ , and a detector with exposure  $\mathcal{E}$  to that part of the sky will collect signal  $\mathcal{S}$  if it is within the solid angle  $\Omega$  of the (magnetically spreading) jet, where

$$\mathcal{S} = \frac{W\mathcal{E}}{\Omega r^2 t} = \frac{W\mathcal{E}}{\Omega r^2 \alpha r} = \frac{4\pi I\mathcal{E}}{\alpha c \eta T \Omega r^3}.$$

### Super-GZK analysis:

As in the earlier jet source analysis, suppose  $\Omega = \Omega_J$ , i.e., the super-GZK particles retain their directions enough that the emission angle of the jet is approximately the solid angle of the jet's flux at the distance of Earth. Suppose  $K$  particles from the source are required for a signal-limited (super-GZK) detection. The distance  $r_K$  is the distance at which a source is expected to produce that many signal events in a detector with exposure  $\mathcal{E}$ :

$$S = K \Rightarrow r_K^3 = \frac{4\pi I\mathcal{E}}{\alpha c \eta T \Omega_J K}. \quad (28)$$

The expected number of sources within the volume of radius  $r_K$  and exposed to the detector is

$$\mu = f_D \int_0^{r_K} \Omega_J r^2 \eta \tau(r) dr = f_D \int_0^{r_K} \Omega_J \alpha \eta r^3 dr = \frac{f_D}{4} \Omega_J \alpha \eta r_K^4 = \frac{f_D}{4} \Omega_J \alpha \eta \left( \frac{4\pi I\mathcal{E}}{\alpha c \eta T \Omega_J K} \right)^{4/3}.$$

Using the mean separation of fossils that have contributed in time  $T$  (so  $R^3 = 1/(\eta T)$ ), the expected number  $\mu$  is

$$\mu = \frac{f_D}{4} (4\pi)^{4/3} \left( \frac{1}{\Omega_J \alpha c} \right)^{1/3} \left( \frac{I\mathcal{E}}{K} \right)^{4/3} \frac{R}{cT}. \quad (29)$$

As previously, the probability of the nearest source being detectable is greater than  $1/2$  provided  $\mu > \ln(2)$ . For jet bursts,

$$\mu > \ln(2) \Leftrightarrow R > \left( \frac{1}{4\pi} \right)^{4/3} \left( \frac{4\ln 2}{f_D} \right) (\alpha c \Omega_J)^{1/3} \left( \frac{K}{I\mathcal{E}} \right)^{4/3} cT. \quad (30)$$

Making the approximation  $I\mathcal{E} = 0.117N$  reduces this to

$$\boxed{R_{min} = \frac{1.7}{f_D} (\alpha c \Omega_J)^{1/3} \left( \frac{K}{N} \right)^{4/3} cT.} \quad (31)$$

A fiducial calculation can be done using the same adhoc values that were adopted previously:  $cT = 100 \text{ Mpc}$ ,  $\alpha c = 1/(36 \times 100^2)$ ,  $\Omega_J = 0.01 \text{ sr}$ ,  $K = 3$ ,  $N = 50$ ,  $f_D = 1/2$ , giving  $R_{min} = 0.024 \text{ Mpc}$ . This is the mean separation of fossils of bursts that contributed to the cosmic ray density during the accumulation time  $T$ . That is,  $R = \eta T$ . Using again the density of  $0.01 \text{ galaxy/Mpc}$ , the maximum mean time between bursts in each galaxy is given by  $\langle \Delta t \rangle_{min} = 0.01 R^3 T$ , and for  $R = 0.024 \text{ Mpc}$ , this gives  $\langle \Delta t \rangle_{min} = 41 \text{ yrs}$ . For the fiducial parameter values adopted here, the nearest source is likely to be detectable provided the mean interval between bursts in each galaxy is at least 41 years.

### Sub-GZK analysis:

As in the persistent jet sources, the solid angle of the jet is not approximately constant except at the highest energies. For the background-limited analysis, it increases with

distance (cf. Appendix C):  $\Omega = 3\kappa r$ . Substituting this into the expression for the jet burst signal gives

$$\mathcal{S} = \frac{4\pi}{3} \frac{I\mathcal{E}}{\alpha c \eta T \kappa r^4}.$$

As usual, the background is  $I\omega\mathcal{E}$ , and the noise is its square root:  $\mathcal{N} = \sqrt{I\kappa r\mathcal{E}}$  (using  $\omega = \kappa r$ ). The signal-to-noise ratio is therefore

$$\mathcal{S}/\mathcal{N} = \frac{4\pi}{3} \sqrt{\frac{I\mathcal{E}}{\kappa^3}} \frac{1}{\alpha c \eta T r^{9/2}}.$$

Setting a detection threshold  $\mathcal{S}/\mathcal{N} = \Sigma$  gives the maximum distance  $r_\Sigma$  at which this signal-to-noise ratio is expected:

$$\mathcal{S}/\mathcal{N} = \Sigma \quad \Leftrightarrow \quad r_\Sigma^{9/2} = \frac{4\pi}{3} \sqrt{\frac{I\mathcal{E}}{\kappa^3}} \frac{1}{\alpha c \eta T \Sigma}. \quad (32)$$

The mean density of jet bursts at distance  $r$  which are “on” and pointed at us is  $f_D \eta \tau \Omega / 4\pi$ . Using  $\tau = \alpha r$  and  $\Omega = 3\kappa r$ , the expected number within a volume of radius  $r_\Sigma$  and exposed to the detector is

$$\mu = f_D \int_0^{r_\Sigma} 3 \kappa \eta \alpha r^4 dr = \frac{3f_D}{5} \kappa \alpha \eta r_\Sigma^5.$$

Using the foregoing expression for  $r_\Sigma$  and  $\eta T = 1/R^3$ , this becomes

$$\mu = \frac{3f_D}{5} \left(\frac{4\pi}{3}\right)^{10/9} \left(\frac{1}{\kappa}\right)^{2/3} \left(\frac{1}{\alpha c}\right)^{1/9} \frac{1}{cT} (I\mathcal{E})^{5/9} \left(\frac{1}{\Sigma}\right)^{10/9} R^{1/3}. \quad (33)$$

As usual, the nearest “on” source pointing at us is likely to be closer than  $r_\Sigma$  and therefore detectable provided  $\mu > \ln(2)$ . This condition is here equivalent to

$$R > \left(\frac{5\ln 2}{3f_D}\right)^3 \left(\frac{3}{4\pi}\right)^{10/3} \kappa^2 \left(\frac{1}{I\mathcal{E}}\right)^{5/3} (\alpha c)^{1/3} (cT)^3 \Sigma^{10/3}. \quad (34)$$

Substituting  $0.117N$  for  $I\mathcal{E}$ , this becomes

$$\boxed{R_{min} = \frac{0.47}{f_D^3} (\alpha c)^{1/3} \left(\frac{\Sigma^2}{N}\right)^{5/3} \kappa^2 (cT)^3.} \quad (35)$$

A fiducial calculation can be made again with the same parameter values as before:  $\kappa = 0.014 \text{ sr/Mpc}$ ,  $N = 10,000$ ,  $\alpha c = 1/(36 \times 10^2)$ ,  $cT = 1500 \text{ Mpc}$ ,  $\Sigma = 3$ ,  $f_D = 1/2$ . The minimum jet burst fossil separation is needed for likely source detection is  $R_{min} = 1.3 \text{ Mpc}$ . Using  $T \approx 4.5 \times 10^9 \text{ yrs}$ , the minimum mean time between bursts in a single galaxy is  $< \Delta t >_{min} = 0.01 R^3 T = 1.0 \times 10^8 \text{ yrs}$ . In the sub-GZK regime, detecting the nearest source is unlikely unless the mean time between bursts in a single galaxy is at least 100 million years. Note, however, that this calculated minimum time is extremely sensitive to the adopted value for  $cT$ , since it is proportional to  $R^3 T$  and  $R$  itself is proportional to  $(cT)^3$ . A small change in the adopted parameter value for  $cT$  produces a dramatic change in the minimum mean time interval.

## 9 Luminosity function

The basic model used in this paper assumes that cosmic ray sources are all the same in the sense that there is a single luminosity  $Q$  (particles emitted per unit time above the cosmic ray energy of interest). This is surely an overly simple idealization. There must be some distribution of luminosities  $n(> Q)$ , i.e. the spatial density of cosmic ray sources with luminosity greater than  $Q$ .

The luminosity function is constrained by the observed intensity  $I(> E)$  of cosmic rays. The luminosity function must account for the particle density  $\frac{4\pi}{c}I$ :

$$T \int Q \frac{dn}{dQ} dQ = \frac{4\pi}{c}I.$$

After imposing this normalization condition, a hypothetical luminosity function  $n(> Q)$  will generally not have a single parameter that could be used in place of  $R$ . The basic model in this paper has focused on the mean separation  $R$  between identical sources. There is no natural generalization of the  $R$ -analysis for an arbitrary luminosity function.

Given any hypothetical luminosity function  $n(> Q)$ , however, one can evaluate the probability  $P$  that one or more cosmic ray sources is detectable. It is given by  $P = 1 - e^{-\mu}$ , where  $\mu$  is the expected number of detectable sources. Suppose a detector with exposure  $\mathcal{E}$  requires  $K$  super-GZK events from some candidate source (or signal-to-noise  $\Sigma$  for sub-GZK events) to confirm a detection. At any distance  $r$ , there is a minimum  $Q(r)$  that is needed for that:

$$\frac{Q(r)\mathcal{E}}{4\pi r^2} > K \quad or \quad \frac{Q(r)}{4\pi r^{5/2}} \sqrt{\frac{\mathcal{E}}{\kappa I}} > \Sigma$$

for super-GZK or sub-GZK analysis, respectively. The expected number of detectable sources is then

$$\mu = \int_0^\infty 4\pi r^2 n(> Q(r)) dr,$$

which then gives the probability  $P = 1 - e^{-\mu}$  that one or more sources is detectable for the hypothetical luminosity function  $n(> Q)$ .

## 10 Incomplete sky coverage

If there are discrete sources of cosmic rays at all, they presumably surround us. The brightest nearby source could be in any part of the sky. It is important to achieve good exposure to the full celestial sphere. No cosmic ray observatory has so far been built with full-sky exposure, although the Auger Observatory has been designed for that.

The analyses in this paper have used a simplifying approximation that a single-site detector has the same good exposure to a fraction of the sky denoted by  $f_D$ . The southern site of the Auger Observatory, by itself, has good exposure to approximately one quarter of the sky at the most southern declinations, decreasing exposure over half of the sky, and zero exposure to the northernmost quarter of the sky. Adding the complementary exposure from its northern site would yield nearly uniform acceptance to cosmic rays from all parts of the sky [9].

The formulas in this paper become more complicated for the real situation in which exposure  $\mathcal{E}(\vec{u})$  is not constant over the celestial sphere but depends on direction  $\vec{u}$ . (Here  $\vec{u}$  denotes a unit direction vector.) In the case of isotropic persistent sources, for example, the signal is

$$\mathcal{S} = \frac{IR^3}{cTr^2} \mathcal{E}(\vec{u}).$$

(See equation 2.) For the super-GZK analysis, the expected signal is  $K$  at radius  $r_K$  given by

$$r_K^2 = \frac{IR^3}{cTK} \mathcal{E}(\vec{u}).$$

The expected number of detectable sources out to distance  $r_K$ , which now depends on direction  $\vec{u}$ , is

$$\mu = \int \int_0^{r_K(\vec{u})} \frac{1}{R^3} r^2 dr d\Omega = \frac{1}{3} \left( \frac{IR}{cTK} \right)^{3/2} \int (\mathcal{E}(\vec{u}))^{3/2} d\Omega. \quad (36)$$

The nearest source is still more likely to be detectable than not provided  $\mu > \ln(2)$ , which now becomes

$$R > \left( \frac{3\ln 2}{\int (\mathcal{E}(\vec{u}))^{3/2} d\Omega} \right)^{2/3} \frac{cTK}{I}.$$

Similar modifications pertain to the sub-GZK analysis. Both the signal and the noise depend on direction  $\vec{u}$ , and the ratio is

$$\mathcal{S}/\mathcal{N} = \sqrt{\frac{I\mathcal{E}(\vec{u})}{\kappa}} \frac{R^3}{cTr^{5/2}}.$$

The distance at which this has a prescribed value  $\Sigma$  is given by

$$r_\Sigma^{5/2} = \sqrt{\frac{I}{\kappa}} \frac{R^3}{cT\Sigma} \sqrt{\mathcal{E}(\vec{u})}.$$

The expected number of detectable sources is

$$\mu = \int \int_0^{r_\Sigma(\vec{u})} \frac{1}{R^3} r^2 dr d\Omega = \frac{1}{3} \left( \frac{IR}{\kappa} \right)^{3/5} \left( \frac{1}{cT\Sigma} \right)^{6/5} \int (\mathcal{E}(\vec{u}))^{3/5} d\Omega. \quad (37)$$

The condition  $\mu > \ln(2)$  becomes

$$R > (3 \ln 2)^{5/3} \frac{\kappa}{I} (cT\Sigma)^2 \left( \frac{1}{\int (\mathcal{E}(\vec{u}))^{3/5} d\Omega} \right)^{5/3}.$$

A simple schematic approximation for the Auger South exposure is  $\mathcal{E} = \mathcal{E}_0 g(v)$  where  $v \equiv \sin(\text{declination})$ ,  $\mathcal{E}_0$  is the rich exposure in the southern sky, and  $g(v)$  is the simple function

$$g(v) \equiv \begin{cases} 1 & \text{if } -1 < v < -1/2 \\ 1/2 - v & \text{if } -1/2 < v < 1/2 \\ 0 & \text{if } v > 1/2. \end{cases}$$

(Note that  $g(v) + g(-v) = 1$ , so identical sites in the north and south provide uniform sky coverage  $\mathcal{E}_0$  in this approximation.)

This schematic model of exposure for Auger South allows an approximate analytic evaluation of  $\int (\mathcal{E}(\vec{u}))^\gamma d\Omega$  for the various powers  $\gamma$  which would arise in the different models considered in this paper. The resulting formulas are certainly less transparent, however, than for the ideal detector with uniform exposure.

An effective sky coverage  $f_D$  for Auger South can be defined by requiring that the expression for  $\mu$  in equation 36 be a scaled version of the expression in equation 4, so

$$\frac{4\pi}{3} \left( \frac{I\mathcal{E}_0 R}{cTK} \right)^{3/2} f_D = \frac{1}{3} \left( \frac{IR}{cTK} \right)^{3/2} 2\pi \int (\mathcal{E}(v))^{3/2} dv.$$

This yields

$$f_D = \frac{2\pi}{4\pi} \int_{-1}^{+1} [g(v)]^{3/2} dv = \frac{9}{20}.$$

For the sub-GZK case, the analogous condition on  $f_D$  coming from equations 37 and 8 leads to

$$f_D = \frac{2\pi}{4\pi} \int_{-1}^{+1} [g(v)]^{3/5} dv = \frac{9}{16}.$$

These results suggest that 1/2 is a suitable estimate for the effective sky coverage  $f_D$  that pertains to a single-site observatory.

## 11 Summary and conclusions

The likelihood of detecting a discrete source of ultra-high energy cosmic rays depends on many variables with unknown values. It has here been assumed that cosmic rays are produced in sources that are randomly distributed throughout the universe with some mean separation  $R$ . The analyses have focused on the question, “What condition on the mean separation ensures that the detection of the nearest source has more than a 50% chance of being detectable?” The question is simplified by assuming that all sources have the same luminosity  $Q$ . The answer certainly depends on the amount of exposure  $\mathcal{E}$  that the detector has to the sources. Since the number of arrival directions in a data set increases in proportion to the detector’s celestial exposure, the answer can be regarded alternatively as depending on the number  $N$  of arrival directions collected above the chosen energy cut. It also depends on the cosmic ray accumulation time  $T$  for that energy cut. Another variable that affects the answer is the statistical significance required for detection – the required number of events  $K$  in the signal-limited case (super-GZK) or the required signal-to-noise ratio  $\Sigma \equiv \mathcal{S}/\mathcal{N}$  in the background-limited case (sub-GZK). In addition, the answer depends on whether the sources are permanent sources emitting isotropically, permanent sources with beamed emission, isotropic bursts, or jet bursts. Magnetic fields also have an important impact on the answer for the background-limited cases.

Properties of intergalactic magnetic fields are not well established. For permanent sources, the relevant information about magnetic fields is summarized by the coefficient  $\kappa$  which governs how the (Gaussian) solid angle of arrival directions grows with distance

from a source:  $\omega = \kappa r$ . Appendix B examines how  $\kappa$  can be calculated from the mean field strength and coherence length in a simple model.

For transient (burst) sources, magnetic smearing of arrival times is relevant as well as the smearing of arrival directions. The time smearing is encapsulated in the coefficient  $\alpha$  by  $\tau = \alpha r$ . Appendix C shows how to estimate  $\alpha$  from the magnetic field properties in the simple model that is used to estimate  $\kappa$ .

The tables below collect the boxed formulas appearing in the text. For each class of models, there are two formulas for the minimum mean separation  $R_{min}$  such that the probability of detection is greater than 1/2. One formula pertains to the super-GZK (signal-limited) regime, and the other formula pertains to the sub-GZK (background-limited) regime.

Table 1: Formulas for  $R_{min}$ . This is the minimum mean source separation for which one expects the nearest source to be detectable. Formulas are tabulated for the signal-limited (super-GZK) case and the background-limited (sub-GZK) cases. These formulas are highlighted by boxes in the text. For a full-sky observatory with uniform exposure,  $N$  should be half the number of arrival directions in the data set of events above the energy corresponding to the cosmic ray accumulation time  $T$ , and the fraction of sky exposed to the detector is  $f_D = 1$ . For a single site like Auger South,  $N$  is the total number of events in the data set, and  $f_D \approx 1/2$ . For non-uniform exposure, it would be better to replace  $N$  by  $I\mathcal{E}/0.117$  and use the exposure  $\mathcal{E}$  that pertains to the target source celestial position and the chosen energy cut for which the cosmic ray intensity is  $I$ .

Source Type	Super-GZK	Sub-GZK
Steady isotropic	$R_{min} = 4.1 \frac{K}{N} cT$	$R_{min} = 1.4 (cT \Sigma)^2 \frac{\kappa}{N}$
Steady Jets	$R_{min} = 4.1 \left(\frac{\Omega_J}{4\pi}\right)^{1/3} \frac{K}{N} cT$	$R_{min} = 1.6 \left(\frac{\Sigma^2}{N}\right)^{4/3} (cT)^{8/3} \kappa^{5/3}$
Isotropic Bursts	$R_{min} = \frac{3.9}{f_D} (\alpha c)^{1/3} \left(\frac{K}{N}\right)^{4/3} cT$	$R_{min} = 0.51 \left(\frac{cT}{f_D}\right)^{7/3} (\alpha c)^{1/3} \left(\frac{\kappa \Sigma^2}{N}\right)^{4/3}$
Jet Bursts	$R_{min} = \frac{1.7}{f_D} (\alpha c \Omega_J)^{1/3} \left(\frac{K}{N}\right)^{4/3} cT$	$R_{min} = \frac{0.47}{f_D^3} (\alpha c)^{1/3} \left(\frac{\Sigma^2}{N}\right)^{5/3} \kappa^2 (cT)^3$

A statistical detection of multiple sources can be expected prior to the detection of any one discrete source. Study of the two-point correlation function (or, equivalently the angular power spectrum) for ultra-high-energy cosmic ray arrival directions can exhibit evidence for many poor clusters of arrival directions even if there is not any one rich cluster of arrival directions that is individually detectable [10]. Evidence for an autocorrelation in the AGASA data was published [11]. Moreover, a correlation of arrival directions with a catalog of candidate sources can supply evidence of discrete sources even if there is no statistical evidence for clustering of the cosmic rays themselves. Exploratory searches have also produced evidence for correlations of that type [12]. These specific claims will be thoroughly tested using larger data sets obtained with new observatories. The failure to detect any individual discrete source so far suggests that there is no really bright cosmic ray source in the sky. It is therefore likely that sources will show up collectively in one of these ways before any individual source becomes obvious.

Whether or not the brightest cosmic ray source will soon become detectable depends on many unknown parameter values. The issue must be decided observationally. Fiducial



estimates show that it is a close call. The answer can go either way, depending on assumptions about the unknown parameter values and the nature of the sources (e.g. bursts, jets). Present detector exposures are already obtaining important constraints on the unknown astrophysical parameters. The viable parameter space will shrink rapidly as the exposure increases. The search will be especially rewarding, however, if it leads to the study of one or more sources and the intervening magnetic fields.

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## 13 Appendix A: Minimum and maximum $R$

The analysis in the text focuses on the mean separation  $R$  between identical sources (density of sources  $n = 1/R^3$ ). The nearest source is unlikely to be detectable if  $R$  is too small, because then each source is too weak. For super-GZK analysis, there is an upper limit  $R_{max}$  as well as lower limit  $R_{min}$  stemming from the fact that the nearest source is unlikely to be much closer to us than the mean separation  $R$ , so  $R$  cannot be much greater than  $cT$ , where  $cT$  is the effective survival distance for those super-GZK particles.

In a volume of radius  $r$  around us, the number of detectable sources is

$$\mu = \frac{4\pi f_D}{3} r^3 / R^3. \quad (38)$$

As explained in section 3, the nearest source is likely to be detectable provided  $\mu > \ln 2$ . The critical value  $\mu = \ln 2$  gives  $R_{min}$  and  $R_{max}$ . Equation 38 shows that these are simply proportional to the corresponding volume radii:

$$R_{min} = \lambda r_{min} \quad \text{and} \quad R_{max} = \lambda r_{max},$$

where the proportionality constant is

$$\lambda = \left( \frac{4\pi f_D}{3 \ln 2} \right)^{1/3}.$$

Equation 1 gives the luminosity  $Q = \frac{4\pi I R^3}{cT}$  per source with mean separation  $R$  which collectively account for the cosmic ray density  $4\pi I/c$ . The signal  $\mathcal{S}$  seen at distance  $r$  after exposure  $\mathcal{E}$  in the basic (isotropic) model is

$$\mathcal{S} = \frac{I \mathcal{E} R^3}{cT r^2} e^{-r/cT},$$

where the GZK attenuation factor  $e^{-r/cT}$  is here included. If  $K$  events are required for a source detection, the critical case is obtained by setting  $\mathcal{S} = K$ . Measure distances

relative to  $cT$ , so  $r = \xi cT$  and  $R = \lambda \xi cT$  when the distance  $r$  is  $r_K = r_{min}$  or  $r_K = r_{max}$ . Then the equation above for  $\mathcal{S}$  reduces to

$$\xi e^{-\xi} = \frac{K}{\lambda^3 I \mathcal{E}}. \quad (39)$$

A solution for  $\xi$  requires  $\frac{K}{\lambda^3 I \mathcal{E}} \leq \frac{1}{e}$ , and there are two solutions except if the equality holds. One solution has  $\xi < 1$  (i.e.  $r_K < cT$ ), and the other has  $\xi > 1$  (i.e.  $r_K > cT$ ). For cases in which  $\frac{K}{\lambda^3 I \mathcal{E}} \ll 1/e$ , then  $e^{-\xi} \equiv e^{-r_K/cT} \cong 1$ . For  $r < r_K$ , the attenuation factor  $e^{-r_K/cT}$  is then very close to 1, and the simpler analysis following equation 2 is fully justified.

The approximation  $I \mathcal{E} = 0.117N$  introduced in section 3 can be used here. Adopting  $f_D = 1/2$  as in the fiducial calculations, one gets

$$\frac{K}{\lambda^3 I \mathcal{E}} < 1/e \Leftrightarrow K < 0.13N.$$

Therefore, provided the data set has more than 8 times the number of arrival directions  $K$  needed for detection of the discrete source, the analysis in the text (omitting the  $e^{-r/cT}$  attenuation factor) is adequate.

The upper limit  $R_{max} = \lambda \xi cT$  is given by the other solution of equation 39 for which  $\xi > 1$ .

## 14 Appendix B: Blurring by magnetic fields

The Larmor radius  $\rho$  characterizes the bending of charged particle trajectories by magnetic fields. For relativistic particles, it is given by

$$\rho_{cm} = \frac{E_{eV}}{300 Z B_G} \Rightarrow \begin{cases} \rho_{Mpc} \doteq \frac{E_{EeV}}{Z B_{nG}} & \text{for extragalactic applications;} \\ \rho_{kpc} \doteq \frac{E_{EeV}}{Z B_{\mu G}} & \text{for galactic applications.} \end{cases}$$

The approximate equations on the right follow from the exact equation on the left using the relations:  $1 EeV \equiv 10^{18} eV$ ,  $1 nG \equiv 10^{-9} G$ ,  $1 Mpc \doteq 3.1 \times 10^{24} cm$ ,  $1 \mu G \equiv 10^{-6} G$ ,  $1 kpc \doteq 3.1 \times 10^{21} cm$ . The particle's electric charge  $Z$  is in units of 1 proton charge. In traveling distance  $D$  through a perpendicular B-field, a particle's trajectory is bent by the angle  $\theta = D/\rho$  in radians.

The Galaxy has a regular magnetic field which tends to follow the spiral arms, and also superposed irregular fields which change direction over short distances along any path. Particles arriving from a distant point source will be systematically deflected by the regular field. An estimate is that they will encounter, on average, a perpendicular magnetic field of about  $2 \mu G$  acting over a path of roughly  $1 kpc$ . Trajectories from a single source would then be systematically bent through the angle

$$\theta = \frac{1 kpc}{\rho} = \frac{(1 kpc)Z(2 \mu G)}{E_{EeV}} \text{ radians.}$$

For protons ( $Z = 1$ ) this is about  $12^\circ$  at 10  $EeV$  and  $1.2^\circ$  at 100  $EeV$ .

A uniform field acting along the entire path from source to detector would cause the arrival direction to differ from the source direction by only  $\theta/2$ . This is because, in that case, the particle does not start from the source in our direction; its initial direction also differs from our line of sight by half of the trajectory bending angle ( $\theta/2$ ). For a distant extragalactic source, however, the regular field changes the arrival direction by the full angle  $\theta$  from the particle's direction of entry into the galaxy (which is the direction from detector to source if the source is very distant and the particle travels on a straight line until reaching the Galaxy).

All protons of one energy are deflected the same amount by the regular magnetic field. With arrival directions of two or more protons of measured energies from the same source, one can determine the product  $BD$  and the source direction. Here  $BD$  is the transverse magnetic field integrated over its range along the incoming trajectory. The regular field produces an arc of arrival directions on the sky ending at the source direction ( $E = \infty$ ) with  $E$ -dependent angular offsets of  $\theta = BD/E$ .

Because of the steep energy spectrum, it can be expected that roughly 3/4 of the arriving particles will have energy less than twice the analysis threshold energy. Using the above estimate of deflection by the regular galactic magnetic field, one would expect that above 10  $EeV$ , for example, the regular field should spread 75% of the arrival directions along an arc of roughly 6 degrees (plus or minus 3 degrees), with the center of that arc displaced from the source direction by approximately 9 degrees. Above 100  $EeV$ , the arc would be roughly 0.6 degree (plus or minus 0.3 degree) with its center displaced from the source by approximately 0.9 degree. The input values for these estimates (2  $\mu G$  for the transverse regular field strength and 1  $kpc$  for the effective path length) are crude estimates, and actual values depend critically on the direction to any given source. This rough estimated does indicate, however, that clusters of arrival directions should not be destroyed by the Galaxy's regular magnetic field for most source directions.

Irregular magnetic fields are the other concern in charged particle astronomy. These are fields that do not have a consistent direction over any particle's trajectory. The particle's direction is continuously being deflected by small magnetic bends that cause it to meander in a random-walk manner. The result is formally the same as multiple coulomb scattering of energetic charged particles in matter. A uni-directional initial beam becomes a Gaussian distribution of particle directions centered on the undeflected direction. The width of the Gaussian distribution ( $\sigma$ ) increases in proportion to the square root of the path length  $r$ . Appendix C shows that the effective solid angle is ( $\omega = 4\pi\sigma^2$ ), which increases in proportion to the path length. The proportionality constant  $\kappa$  is defined by  $\omega = \kappa r$ .

A simple model of magnetic blurring by irregular fields is that a particle encounters a different field orientation in each segment of length  $L$  along its path. The field has a mean strength  $B$  with a random orientation which is constant over each segment. Consider deflection in any plane containing the original direction. The B-component perpendicular to that plane is expected to be  $B/\sqrt{3}$ . Over a segment length  $L$  the deflection in the plane is

$$\theta_s = \frac{LZB}{\sqrt{3}E}.$$

This is the random walk step size. After  $n := r/L$  steps, the directions in that plane are distributed with a Gaussian of width

$$\sigma = \sqrt{n}\theta_s = \sqrt{\frac{r}{L} \frac{LZB}{\sqrt{3}E}} = \sqrt{\frac{rL}{3}} \frac{ZB}{E}.$$

This is the distribution of directions relative to an original beam direction. What is relevant is the direction of a detected particle relative to the direction from the detector to the source. That direction to the source does not correspond to the original particle direction because the particle will have been displaced laterally from the beam. The lateral displacement is correlated with the offset of the arrival direction. As in multiple Coulomb scattering, the arrival directions are distributed around the direction to the beam origin with a Gaussian  $\sigma$  which is smaller by  $1/\sqrt{3}$ . This is a straightforward consequence of the statistical correlation of angular offset with spatial offset in the random walk process. The direction back to the source is different for particles that have a net deflection to the left than for those that have a net deflection to the right, and the final direction is statistically correlated with the net deflection.

Using  $\kappa = \omega/r$  together with  $\omega = 4\pi\sigma^2$  and  $\sigma = \sqrt{\frac{r}{L}} \frac{\theta_s}{\sqrt{3}}$  yields a formula for  $\kappa$ :

$$\kappa = \frac{4\pi}{9} \frac{LZ^2B^2}{E^2}.$$

This expression for  $\kappa$  is used in the text for evaluating the magnetic blurring due to random intergalactic magnetic fields between a distant source and the Earth. There is also a contribution by irregular magnetic fields within the Galaxy, and this expression for  $\kappa$  can be used to estimate its magnitude. Suppose  $B = 3 \mu G$  for the randomly-oriented field strength, that  $L = 0.1 \text{ kpc}$ , and the travel path through those irregular galactic fields has length  $r = 1 \text{ kpc}$ . Then the Gaussian spread  $\sigma$  is given (for  $Z = 1$ ) in radians by

$$\sigma = \sqrt{\omega/4\pi} = \sqrt{\frac{LZ^2B^2r}{9E^2}} = \frac{0.32}{E}.$$

Setting  $E = 10 \text{ EeV}$  gives  $\sigma = 1.8^\circ$ .

## 15 Appendix C: Time spreading by magnetic fields

The simple model of irregular fields used in Appendix B provides an estimate for the coefficient  $\alpha$  that governs the time spread  $\tau$  for a source at distance  $r$  by  $\tau = \alpha r$ . The estimate for  $\tau$  is based on the expected difference in transit time for a charged particle of energy  $E$  compared to an undeflected neutral speed-of-light particle. There is a time difference in each path segment of length  $L = \rho\theta$  because the curved trajectory is longer than the straight line distance between the endpoints. (Here  $\rho$  is the Larmor radius based on the perpendicular magnetic field, and theta is the trajectory bending angle while traveling distance  $L$ .) The time difference is

$$\Delta t = \frac{1}{c}(L - 2\rho\sin(\theta/2)) = \frac{\rho}{c}(\theta - 2\sin(\theta/2)) \approx \frac{\rho}{c} \frac{\theta^3}{24} = \frac{L^3}{24c\rho^2}.$$

The total time difference is then

$$\tau = \frac{r}{L} \Delta t = \frac{L^2}{24R^2} \frac{r}{c}.$$

Using  $R = E/ZB_{\perp}$  and  $\langle B_{\perp}^2 \rangle = \frac{2}{3}B^2$ , this gives the coefficient  $\alpha \equiv \tau/r$ :

$$\alpha = \frac{L^2 Z^2 B^2}{36cE^2}.$$

Here  $B$  is the magnetic field strength whose direction is randomly oriented along each trajectory segment of length  $L$ .

This estimate ignores a second-order contribution to  $\tau$  due to the segments themselves meandering about the straight line from the source to the arrival point.

## 16 Appendix D: Signal and noise with Gaussian-distributed arrival directions

Random intergalactic magnetic fields are expected to produce a Gaussian distribution of arrival directions at the detector, centered on the source direction. Suppose an observed source has arrival directions distributed about a central direction with a Gaussian of width  $\sigma$ . This means in any one dimension the probability distribution for offset  $\theta_x$  is  $P(\theta_x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\theta_x^2/2\sigma^2)$ , and the 2-dimensional (space angle) offset  $\theta = \sqrt{\theta_x^2 + \theta_y^2}$  has probability  $P(\theta) = \frac{1}{2\pi\sigma^2} \exp(-\theta^2/2\sigma^2)$ . When testing a discrete source with an apparent Gaussian distribution of arrival directions, it is sensible to give more weight to arrival directions that are near the center of the distribution and little weight to arrival directions that are far from it. The appropriate weighting function is

$$w = 4\pi\sigma^2 P(\theta) = 2\exp(-\theta^2/2\sigma^2).$$

This choice enjoys three important properties:

- (1) The *shape* of the Gaussian function of width  $\sigma$  maximizes the signal among all possible weighting functions of the same integral normalization.
- (2) With the  $4\pi\sigma^2$  integral normalization, the weighted integral of any uniform background has expected fluctuations given by the square root of the background itself. This expected background is  $4\pi\sigma^2 \times (\text{background density})$ , and the RMS fluctuation in that weighted background integral is its square root.
- (3) With the  $4\pi\sigma^2$  integral normalization, a density  $N_0 P(\theta)$  of smeared-out arrival directions due to  $N_0$  cosmic rays from a discrete source results in a weighted integral equal to  $N_0$ , i.e. the actual number of smeared-out directions.

Defining signal and background as weighted integrals with this weighting function, their difference is the expected number of events producing the signal. Moreover, the

usual  $S/N$  statistical significance pertains with the noise  $\mathcal{N}$  being simply the square root of the background, as in analyses without a weighting function. One can regard  $2\sigma$  as an effective radius, giving  $4\pi\sigma^2$  as an effective collecting area.

A more careful analysis should use Fisher distributions [13] rather than Gaussians for celestial analyses, especially for the broad distributions that are expected in sub-GZK analyses.

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